Mathematical logic.

* Logic is to provide rules by which one can determine whether any particular statement or reasoning is valid

* Logic is concerned with all kinds of reasonings whether they may be legal arguments or mathematical proofs or conclusions in a scientific theory based upon the set of hypothesis

* Any collection of rules or any theory needs a Language in which these rules or theory can be stated there fore necessary first to develop a formal Language called object Language

* A formal Language is one in which the syntax is well-defined

* we use symbols in the object Language so it is called Symbolic Language

* Natural Language (English) will be called meta Language

* only declarative sentences will be admitted in the object language, which have two possible values called "truth values".

The two truth values are true and false. They are denoted by T or F. Sometimes it will be denoted by symbols 1 and 0.

Examples of sentences:

1. Canada is a Country
2. Moscow is the Capital of Spain
3. This Statement is false
4. 1 + 101 = 110
5. Close the door.

Declarative sentences in the object language are of two types :.

The first type includes those sentences which are considered to be primitive in the object language. It is denoted by distinct symbols A, B, C . . . . , P, Q . . . .

The second type are obtained from the Primitive ones by using certain symbols, called Connectives and certain Punctuation marks, such as paranthesis to join primitive sentences.

All the declarative sentences to which is possible to assign one and only one of the two possible truth values are called Statements

These statements which do not contain any of the Connectives are called atomic (Primary, Primitive) Statements.

Connectives:

It is possible to construct rather complicated statements from simpler statements by using certain connecting words or expressions known as "Sentential Connectives".

The statements that we consider initially are simple statements called atomic or Primary Statements.

New Statements can be formed from atomic statements through the use of sentential connectives. The resulting statements are called molecular or Compound statements

⑧ The atomic statements are called molecular or Compound Statements.

As an illustration. let

P: It is raning today

Q: It is showing

and let R be a statement variable whose possible replacements are P and Q. If no replacement of R is specified. it remains a statement variable and has no truth values.

1. Negation:

The negation of a Statement is generally formed by introducing the word "not" at a proper place in the statement or by prefixing the Statement phrase "It is not the Case that".

If P denotes a statement then the negation of P is written as "$\neg P$" and read as "not P"

If truth value of P is T then the truth value of $\neg P$ is F.

Ex: 1

Consider the statement

P : Chennai is a city. Then ¬P is the statement

¬P : Chennai is not a city.

Ex: 2

Consider the statement

P: I went to my class yesterday

1. ¬P : I did not go to my class yesterday
2. ¬P : I was absent from my class yesterday
3. ¬P : It is not the case that I went to my class yesterday

Alternative symbols used in the literature "~" or "a bar" or "not".

ie, ¬P is written as ~P (or) $\overline{P}$ (or) NOT P.

Negation is unary operation which operates on a single statement or a variable.

Truth table for negation:

P	¬P
T	F
F	T

2. Conjunction :

The Conjunction of two statements p and q is the statement p∧q which is read as "p and q". The statement p and q has the truth value T whenever both P and Q have the truth value T, otherwise it has the truth value F. (5)

Truth Table for Conjunction:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Ex:1 Form the Conjunction of

1 P: It is raining today

2 Q: There are twenty tables in this room

$\Rightarrow$ $P \wedge Q$: It is raining today and there are twenty tables in this room.

Ex:2 Translate into symbolic form from following statement

Jack and Jill went up the hill

Soln:

P: Jack went up the hill

Q: Jill went up the hill

$P \wedge Q$: Jack went up the hill and Jill went up the hill

Ex:

The connective "and" is sometimes used in a different sense and in such cases it cannot be translated by symbol "$\wedge$"

Ex: Consider the statements

1. Roses are red and violets are blue
2. He opened the book and read it
3. Jack and Jill are brothers.

Here in statement 1, the conjunction "and" is used in the same sense as the symbol ($\wedge$).

In statement 2, the word "and" is used in the sense of "and then".

In statement 3 the word "and" is not a conjunction.

Conjunction is symmetric as far as p and Q are concerned.

3. Disjunction: (or)

The disjunction of two statements p and Q is the statement "$p \vee Q$" which is read as "p or Q". The statement $p \vee Q$ has the truth value "F" only when both p and Q have the truth value "F" otherwise it is true.

Truth table for disjunction

P	Q	P∨Q
T	T	T
T	F	T
F	T	T
F	F	F

The Connectives ¬ and ∧ have the same meaning as the words "not" and "and" in general.

But the Connectives "∨" is not always same as the word "or" because the word "or" in English is Commonly used both as an 'exclusive OR' and "Inclusive OR"

Ex: Consider the following Statements

1. I Shall watch the game on TV or go to the game
2. There is Something wrong with the bulb or with the wiring.
3. Twenty or thirty animals were killed in fire Today.

In Statement 1, the Connective or is used in the exclusive or. ie, one or other exist but not both

In 2. the meaning is one or other or both [Inclusive]

In 3. "or" is used for indicating an approximate no of animals (exclusive)

The defn of disjunction states that 'inclusive or' only

Statement Formulae & Truth Tables

- These statements which donot contain any connectives are called atomic or primary or simple statements.

- These statements which contain one or more primary statements and some connectives are called molecular or composite or compound statements

Ex: Let p and Q be any two statements. Some of the compound statements formed by using p and Q are

$\neg P$, $P \vee Q$, $(P \wedge Q) \vee (\neg P)$, $P \wedge (\neg Q)$

In general if there are n-distinct components in a statement formula. we need to consider 2^n possible combinations of truth values in order to obtain the truth table.

→ ① Construct the truth table for the statement formula $P \vee \neg Q$

P	Q	$\neg Q$	$P \vee \neg Q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

② $P \wedge \neg P$

③ $(P \vee Q) \vee \neg P$

P	Q	$P \vee Q$	$\neg P$	$(P \vee Q) \vee \neg P$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

Conditional and Biconditional

If P and Q are any two statements then the Statement $P \rightarrow Q$ which is read as "If p then Q" is called the Conditional Statement.

This statement $P \rightarrow Q$ has a truth value F when Q has the truth value F and the p has the truth value T. Otherwise it has truth value T.

Truth table for Conditional Statement

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$\wedge$ - T - T - T

$\vee$ - F - F - F

$\rightarrow$ - T - T - F

$P \wedge Q$ - T - T - T

$P \vee Q$ - F - F - F

1 mark: The statement P is called anticedent and Q is consequent in $P \rightarrow Q$.

Ex:1 Express in English, the statement $P \rightarrow Q$ where P: The Sun is shining

Q: $4+3>5$

Soln: $P \rightarrow Q$: If the Sun is shining then $4+3>5$

The truth value of $P \rightarrow Q$ is T.

Ex:2 Write the following statement in symbolic form

If either Jerry takes Calculus or Ken takes Statistics then Larry take English.

$(J \vee K) \rightarrow L$

Statement :-

J : Jerry takes Calculus

K : Ken takes Statistics

L : Larry takes English

$\Rightarrow (J \vee K) \rightarrow L$

(10)

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3. Write in Symbolic form from the following statement (U)

The ~~crob~~ will be destroyed, if there is flood.

Soln: crop

The above statement can be written as

If there is flood, then the crob will be destroyed

F : There is flood

C : The crob will be destroyed

$F \to C$.

4. Construct the truth table for

U.Q $(P \to Q) \wedge (Q \to P)$

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \wedge (Q \to P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Biconditional :

If P and Q are any two statements then the statement $P \rightleftarrows Q$ which is read as "p if and only if Q" and abreciated as "p iff Q" is called a Biconditional statement

This statement $P \rightleftarrows Q$ has the truth value T whenever both P and Q have identical truth values

Truth table for Biconditional

P	Q	$P \rightleftarrows Q$
T	T	T
T	F	F
F	T	F
F	F	T

The statement $P \rightleftarrows Q$ is also translated as

"P is necessary and sufficient for Q"

UQ

Ex: Construct the truth table for the formula

$\neg(P \wedge Q) \rightleftarrows (\neg P \overset{\vee}{\wedge} \neg Q)$

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg(P\wedge Q) \rightleftarrows (\neg P \wedge \neg Q)$
T	T	F	F	F	T	F	T
T	F	F	T	F	F	T	F
F	T	T	F	F	F	T	F
F	F	T	T	T	F	T	T

2. Construct the truth table of the following formulae

(a) $(Q \wedge (P \to Q)) \to P$

(b) $\neg(P \vee (Q \wedge R)) \rightleftarrows (P \vee Q) \wedge (P \vee R)$

(a)

P	Q	$P \to Q$	$Q \wedge (P \to Q)$	$(Q \wedge (P \to Q)) \to P$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$\neg P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$	$\neg(P \vee (Q \wedge R)) \rightleftarrows (P \vee Q) \wedge (P \vee R)$

Find the formula using P, Q and the connectives $\wedge, \vee, \neg$ whose truth values are identical to the truth values of $P \bar{\vee} Q$. (13)

P	Q	$P \bar{\vee} Q$	$P \vee Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$(P \vee Q) \wedge \neg(P \wedge Q)$
T	T	F	T	T	F	F
F	T	T	T	F	T	T
T	F	T	T	F	T	T
F	F	F	F	F	T	F

$$P \bar{\vee} Q \Leftrightarrow (P \vee Q) \wedge \neg(P \wedge Q)$$

Well-formed formulae:-

A statement formula is an expression which is a string consisting of variables, parentheses and connective symbols.

Defn: Well-formed formula:

A well-formed formula can be generated by the following rules

1. A statement variable standing alone is well-formed formula.
2. If A is well-formed formula then $\neg A$ is a well-formed formula.
3. If A and B are well formed formulae, then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \rightleftarrows B)$ are well formed formulae.
4. A string of symbols containing the statement variables, connectives and parentheses is a well formed formula iff it can be obtained by finitely many applications of the rules 1, 2 & 3.

3. Construct truth table for

(a) $(P\wedge(P\to Q))\to Q$

P	Q	$P\to Q$	$P\wedge(P\to Q)$	$(P\wedge(P\to Q))\to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

4. The truth values of P & Q as T and those of R & S as F. Find the truth values of the following.

i. $(\neg(P\wedge Q)\vee\neg R)\vee((Q\rightleftarrows\neg P)\to(R\vee\neg S))$

(a) P	Q	R	S	$P\wedge Q$	$\neg(P\wedge Q)$	$\neg R$	$\neg(P\wedge Q)\vee\neg R$	$\neg P$	$Q\rightleftarrows\neg P$	$\neg(P\wedge Q)\vee\neg R \vee(Q\rightleftarrows\neg P)$	$R\vee\neg S$
T	T	F	F	T	F	T	T	F	F	T	T

$(\neg(P\wedge Q)\vee\neg R)\vee((Q\rightleftarrows\neg P)\to(R\vee\neg S))$
T

(b) $(P\rightleftarrows R)\wedge(\neg Q\to S)$

P	Q	R	S	$P\rightleftarrows R$	$\neg Q$	$\neg Q\to S$	$(P\rightleftarrows R)\wedge(\neg Q\to S)$
T	+	F	F	F	F	T	F

$(P\vee(Q\to(R\wedge\neg P)))\rightleftarrows(Q\vee\neg S)$

(c)

P	Q	R	S	$\neg P$	$R\wedge\neg P$	$Q\to R\wedge\neg P$	$P\vee(Q\to(R\vee\neg P)$	$\neg S$	$Q\vee\neg S$
T	T	F	F	F	F	F	T	T	T

$(P\vee(Q\to(R\wedge\neg P))\rightleftarrows(Q\vee\neg S)$
T

A Connective denoted by $\bar{\vee}$ is defined by the following table.

P	Q	$P\bar{\vee}Q$
T	T	F
T	F	T
F	T	T
F	F	F

Ex. For well-formed formulae

$\neg(P \vee Q)$, $\neg(P \wedge Q)$, $(P \to (P \vee Q))$

For not well-formed formulae

1. $\neg P \wedge Q$ $\to$ $(\neg P \wedge Q)$ (or) $\neg(P \wedge Q)$
2. $(P \to Q) \to (\wedge Q)$
3. $(P \to Q) \to Q)$

Tautologies:

(i) Defn: A statement formula which is true regardless of the truth values of the statements which replace the variables in it is called a Universally valid formula (or) A Tautology (or) A logical truth

A statement formula which is false regardless of the truth value of the statements which replace the variable in it is called contradiction.

The negation of a contradiction is a Tautology.

We may say that A statement formula which is a Tautology is identically true and a formula which is a contradiction is identically false.

Contigency:

(ii) * A statement formula that is neither a tautology nor a contradiction is called a Contigency

* The conjunction of two tautologies are also Tautology.

Defn: Substitution Instance:

A formula A is called a Substitution instance of another formula B If A can be obtained from B by substituting formula for some variables of B. with the condition that the same formula is substituted for same variable each time it occurs.

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Ex:

1. let B: $P \to (T \wedge P)$

Now, substitute $P \rightleftarrows Q$ for P in B

A: $(P \rightleftarrows Q) \to (T \wedge (P \rightleftarrows Q))$ (is a substitution instance of B)

C: $(\overset{P \rightleftarrows Q}{\cancel{R \rightleftarrows S}}) \to (T \wedge P)$ is not a substitution instance of B

2. Consider $P \to \neg Q$ the substitution instances are

(i) $(R \wedge \neg S) \to \neg Q$

(ii) $(R \wedge \neg S) \to \neg(R \wedge \neg S)$

(iii) $(R \wedge \neg S) \to \neg P$

(iv) $Q \to \neg(P \wedge \neg Q)$

Result:

(Any substitution instance of a Tautology is a Tautology)

Ex:

Consider the Tautology $P \vee \neg P$

1. $(R \to S) \vee \neg(R \to S)$

R	S	$R \to S$	$\neg(R \to S)$	$(R \to S) \vee \neg(R \to S)$
T	T	T	F	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

2. $((P \vee S) \wedge R) \vee \neg((P \vee S) \wedge R)$

3. $(((P \vee \neg Q) \to R) \rightleftarrows S) \vee \neg(((P \vee \neg Q) \to R) \rightleftarrows S)$

Thus if it is possible to deduct whether a given formula is a substitution instance of a Tautology then it is known that the given formula is also a Tautology

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1. From the formulae given below those which are well formed formula.

   i) $P \to (P \vee Q)$

   ii) $((P \to (Q \to R)) \to ((P \to Q) \to (P \to R)))$ w.f

   iii) $((P \wedge Q) \rightleftarrows P)$ wf

2. Produce the substitution instance of the following formula for the given substitutions

   a. $(((P \to Q) \to P \to P)$ $((P \to Q) \to (P \wedge Q) \to R) \to (P \to Q) \to (P \to Q))$

   Substitute $P \to Q$ for P and $((P \wedge Q) \to R)$ for Q.

   b. $((P \to Q) \to (Q \to P))$ $Q \to (P \wedge \neg P)) \to ((P \wedge \neg P) \to Q)$

   Substitute Q for P and $(P \wedge \neg P)$ for Q.

3. $((P \to (Q \to R)) \to ((P \to Q) \to (P \to R)))$ check whether it is a Tautology or Contradiction. (T).

Equivalence of formulae:

Defn:

let A and B be two statement formula. let $P_1, P_2 \cdots P_n$ denote all the variables occuring in both A and B. Consider an assignment truth values to $P_1 \cdots P_n$ and resulting to the values of A and B. If truth values of A is equal to the truth values of B for every one of 2^n possible sets of truth values assigned to $P_1, P_2 \cdots P_n$ then A and B are said to be equivalent.

Assuming that the variables and the assignment truth values to the variables appears in the truth values of A and B. Then the final columns in the truth table for A & B are equal if A and B are equal.

P $\neg\neg P$ $\neg\neg\neg P$

T F T

$P \vee P$: T

Ex:

1. $\neg\neg P$ is equivalent to P
2. $P \vee P$ is equivalent to P
3. $(P \overset{\wedge}{\vee} \neg P) \vee Q$ is equivalent to Q →
4. $(P \vee \neg P) \overset{\wedge}{\vee} Q$ is equivalent to Q
5. $(P \vee \neg P)$ is equivalent to $Q \vee \neg Q$

P	Q	$\neg P$	$(P \wedge \neg P)$	$\vee Q$
T	T	F	F	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	F

For equivalence of two formulae, it is not necessary to assume that they both contain the same variables.

The statement formula A & B are equivalent provided $A \rightleftarrows B$ is a Tautology. Conversely if $A \rightleftarrows B$ is a Tautology then A & B are equivalent.

Notation:

We shall represent the equivalence of two formula say A & B by writing $A \Leftrightarrow B$ which is read as "A is equivalence to B".

* Equivalence is a symmetric relation

i.e, "$A \Leftrightarrow B$" is the same as "$B \Leftrightarrow A$".

Also $A \Leftrightarrow B$ & $B \Leftrightarrow C$ then $A \Leftrightarrow C$

This relationship is called transitive

UQ Ex: Prove that $(P \to Q) \Leftrightarrow (\neg P \vee Q)$

P	Q	$\neg P$	$\neg P \vee Q$	$P \to Q$	$(P \to Q) \rightleftarrows (\neg P \vee Q)$
T	T	F	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
F	F	T	T	T	T

Equivalent formula:

$\Leftrightarrow$ Equivalent

1.	$P \vee P \Leftrightarrow P$	$P \Leftrightarrow P \wedge P$ [Idempotent Law
2.	$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$	$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$ [Associative law
3.	$(P \vee Q) \Leftrightarrow (Q \vee P)$	$(P \wedge Q) \Leftrightarrow (Q \wedge P)$ [Commutative
4.	$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ [Distributive]
5.	$P \vee F \Leftrightarrow P$	$P \wedge T \Leftrightarrow P$
6.	$P \vee T \Leftrightarrow T$	$P \wedge F \Leftrightarrow F$
7.	$P \vee \neg P \Leftrightarrow T$	$P \wedge \neg P \Leftrightarrow F$
8.	$P \vee (P \wedge Q) \Leftrightarrow P$	$P \wedge (P \vee Q) \Leftrightarrow P$ [Absorption Law]
9.	$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$ [De morgan's]

In the above table:

The pairs of equivalent formulae are arranged two to a line such as $A_1 \Leftrightarrow B_1$ & $A_2 \Leftrightarrow B_2$ for each pair of A_1, B_1 there is corresponding A_2, B_2 in which '$\vee$' is replaced by '$\wedge$' & '$\wedge$' is replaced by '$\vee$' and T by F and F by T.

Then A_1 & A_2 are said to be duals of each other and so are B_1 & B_2.

Ex: 2 S.T $P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R) \Leftrightarrow (P \wedge Q) \to R$

We already known that

$(Q \to R) \Leftrightarrow (\neg Q \vee R)$

$P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R)$ → ex 1

$\Leftrightarrow \neg P \vee (\neg Q \vee R)$

$\Leftrightarrow (\neg P \vee \neg Q) \vee R$ [Associative law]

$\Leftrightarrow \neg(P \wedge Q) \vee R$ [De'morgan's law]

$\Leftrightarrow (P \wedge Q) \to R$

$P \to Q$ → $\neg Q \vee R$ → (if) $\Leftrightarrow$ Equivalent

Ex:3

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$

Q) $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

$\Leftrightarrow (\neg P \wedge (\neg Q \wedge R)) \vee (Q \vee P) \wedge R$ [Distributive law]

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R$ [Associative]

$\Leftrightarrow ((\neg P \wedge \neg Q) \vee (Q \vee P)) \wedge R$ [Distributive]

$\Leftrightarrow (\neg(P \vee Q) \vee (P \vee Q)) \wedge R$ [De'morgan's law]

$\Leftrightarrow T \wedge R$ [$P \vee \neg P \Leftrightarrow T$]

$\Leftrightarrow R$ (from rule 5) $\Rightarrow P \wedge T = P$

Ex:4

P.T $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a Tautology.

Sol:

$(\neg P \wedge \neg Q) \Leftrightarrow \neg(P \vee Q)$ [by De'morgans law]

$(\neg P \wedge \neg R) \Leftrightarrow \neg(P \vee R)$

$(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R) \Leftrightarrow \neg(P \vee Q) \vee \neg(P \vee R)$.

$\Leftrightarrow \neg((P \vee Q) \wedge (P \vee R))$ [by De'morgans law]

$\neg(\neg P \wedge (\neg Q \vee \neg R)) \Leftrightarrow \neg(\neg P \wedge \neg(Q \wedge R))$ [De'morgan's law]

$\Leftrightarrow \neg\neg(P \vee (Q \wedge R))$ [De'morgan's law]

$\Leftrightarrow P \vee (Q \wedge R)$

$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$ [Distributive law]

Now

$((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$

$\Leftrightarrow ((P \vee Q) \wedge (P \vee Q) \wedge (P \vee R)) \vee \neg((P \vee Q) \wedge (P \vee R))$

$\Leftrightarrow ((P \vee Q) \wedge (P \vee R)) \vee \neg((P \vee Q) \wedge (P \vee R))$

$\Leftrightarrow T$ [$P \vee \neg P \Leftrightarrow T$]

The above formulae is a substitution instance of $P \vee \neg P$

Duality Law:

Two formula A and A^* are said to be duals of each other if either one can obtained from the other formula by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$

The connectives $\wedge$ and $\vee$ are also called duals of each other.

If formula A contains the special variables T or F then A^* its dual is obtained by replacing T by F and F by T in addition to the above mentioned interchanges.

1 mark

Ex:1 Write the duals of (a) $(P \vee Q) \wedge R$ (b) $(P \wedge Q) \vee T$

(c) $\neg(P \vee Q) \wedge (P \vee \neg(Q \wedge \neg S))$

Soln:

(a) $(P \wedge Q) \vee R$ (b) $(P \vee Q) \wedge F$ (c) $\neg(P \wedge Q) \vee (P \wedge \neg(Q \vee \neg S))$

Theorem: 1.21

Let A and A^* be dual formulae and let $P_1, \dots P_n$ be all the atomic variables that occur in A & A^*.

a, we may write A as $A(P_1, P_2 \cdots P_n)$ and A^* as $A^*(P_1, P_2 \cdots P_n)$ then through the use of De'morgan's laws

$$P \wedge Q \Leftrightarrow \neg(\neg P \vee \neg Q)$$
$$P \vee Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$$

$$\neg A(P_1, P_2 \cdots P_n) \Leftrightarrow A^*(\neg P_1, \neg P_2 \cdots \neg P_n) \quad \text{——} ①$$

Thus the negation of a formula is equivalent to its dual in which every variable is replaced by its negation

$$A(\neg P_1, \neg P_2 \cdots \neg P_n) \Leftrightarrow \neg A^*(P_1, P_2 \cdots P_n) \quad \text{——} ②$$

Exis $\neg(\neg P \wedge (\neg Q \wedge P)) \ldots$

Ex:

verify the equivalence

$$\left(\neg A(P_1, P_2 \ldots P_n) \Leftrightarrow A^*(\neg P_1, \neg P_2 \ldots \neg P_n)\right)$$

if $A(P, Q, R) = \neg P \wedge \neg(Q \vee R)$ } dual form

$A^*(P, Q, R) = \neg P \vee \neg(Q \wedge R)$

$A^*(\neg P, \neg Q, \neg R) = \neg(\neg P) \vee \neg(\neg Q \wedge \neg R)$

$= P \vee \neg\neg(Q \vee R)$

$= P \vee (Q \vee R)$ ——— (1)

$A(P, Q, R) = \neg P \wedge \neg(Q \vee R).$

$\neg A(P, Q, R) = \neg(\neg P \wedge \neg(Q \vee R))$

$= \neg\neg(P \vee (Q \vee R))$

$= P \vee (Q \vee R)$ ——— (2)

from (1) & (2)

$$\neg A(P_1, P_2 \ldots P_n) \Leftrightarrow A^*(\neg P_1, \neg P_2 \ldots \neg P_n)$$

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Theorem: 1.22

If any two formulae are equivalent then their duals are equivalent to each other. Inotherwords if $A \Leftrightarrow B$ then $A^* \Leftrightarrow B^*$

Proof:

Let $P_1, P_2 \ldots P_n$ be all the atomic variables appearing in the formulae A & B

Given then $A \Leftrightarrow B$ means "$A \rightleftarrows B$" is a Tautology then the following are Tautologies

$A(P_1, P_2 \ldots P_n) \rightleftarrows B(P_1, P_2 \ldots P_n)$ ——— (1)

$A(\neg P_1, \neg P_2 \ldots \neg P_n) \rightleftarrows B(\neg P_1, \neg P_2 \ldots \neg P_n)$ ——— (2)

Using this relation w.r.t. $A(\neg P_1, \neg P_2 \ldots \neg P_n) \Leftrightarrow \neg A^*(P_1, P_2 \ldots P_n)$

$A(\neg P_1, \neg P_2 \ldots \neg P_n) \Leftrightarrow \neg A^*(P_1, P_2 \ldots P_n)$ (3) similarly $B(\neg P_1, \neg P_2 \ldots \neg P_n) \Leftrightarrow \neg B^*(P_1 \ldots)$ (4)

from (3), (4) & (6) we have $A(\neg P_1, \neg P_2 \ldots \neg P_n) \Leftrightarrow B(\neg P_1, \neg P_2 \ldots)$

$\neg A^*(P_1, P_2 \ldots P_n) \rightleftarrows \neg B^*(P_1, P_2 \ldots P_n)$

$\therefore A^* \Leftrightarrow B^*.$

$\neg B^*(P_1, P_2 \ldots P_n) \Leftrightarrow B(\neg P_1, \neg P_2 \ldots \neg P_n)$

Ex-2 (a) $\neg(P\wedge Q) \to (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (\neg P \vee Q)$

(1.Q) ~~$P \to Q \Leftrightarrow \neg P \vee Q$~~ (b) $(P\vee Q)\wedge(\neg P\wedge(\neg P\wedge Q)) \Leftrightarrow (\neg P\wedge Q)$

Proof:

(a) $\neg(P\wedge Q) \to (\neg P\vee(\neg P\vee Q)) \Leftrightarrow \neg\neg(P\wedge Q) \vee (\neg P\vee(\neg P\vee Q))$

$\Leftrightarrow (P\wedge Q)\vee((\neg P\vee\neg P)\vee Q)$ [Associative law]

$\Leftrightarrow (P\wedge Q)\vee(\neg P\vee Q)$ [by $P\vee P \Leftrightarrow P$

$\Leftrightarrow ((P\wedge Q)\vee\neg P)\vee Q$ [Associative

$\Leftrightarrow ((P\vee\neg P)\wedge(Q\vee\neg P))\vee Q$ [Distributive.

$\Leftrightarrow (T\wedge(Q\vee\neg P))\vee Q$ [by $P\vee\neg P \Leftrightarrow T$]

$\boxed{P\wedge T = P}$

$\Leftrightarrow (Q\vee\neg P)\vee Q$

$\Leftrightarrow (\neg P\vee Q)\vee Q$

$\Leftrightarrow \neg P\vee(Q\vee Q)$ [Associative]

$\Leftrightarrow \neg P\vee Q$. [by $P\vee P\Leftrightarrow P$]

(b) $(P\wedge Q)\vee(\neg P\vee(\neg P\vee Q)) \Leftrightarrow (\neg P\vee Q)$

Now writing the duals we have

$(P\vee Q)\wedge(\neg P\wedge(\neg P\wedge Q)) \Leftrightarrow (\neg P\wedge Q)$

b. $(P\vee Q)\wedge(\neg P\wedge(\neg P\wedge Q)) \Leftrightarrow (\neg P\wedge Q)$

From (a)

$(P\wedge Q)\vee(\neg P\vee(\neg P\vee Q)) \Leftrightarrow (\neg P\vee Q)$

Now writing the duals we have,

$(P\vee Q)\wedge(\neg P\wedge(\neg P\wedge Q)) \Leftrightarrow (\neg P\wedge Q)$

By thm 1.2Q [write the statement here if it is big question means prove that]

Ex:
$1 \ni 7 / 1 \neg P \wedge$ /-

Tautological Implications :

The Connectives $\wedge, \vee, \& \rightleftarrows$ are Symmetric in the sense that

$$P \wedge Q \Leftrightarrow Q \wedge P$$

$$P \vee Q \Leftrightarrow Q \vee P$$

$$P \rightleftarrows Q \Leftrightarrow Q \rightleftarrows P$$

But $P \to Q$ is not equivalent to $Q \to P$

For any statement formula $P \to Q$. the statement formula $Q \to P$ is called its converse.

$\neg P \to \neg Q$ is called its inverse and

$\neg Q \to \neg P$ is called its contrapositive

Defn. : Tautologically Imply:

A statement 'A' is said to be tautologically imply a statement 'B' iff '$A \to B$' is a tautology.

We shall denote this by '$A \Rightarrow B$' which read as "A implies B" $A \Rightarrow B$ states that $A \to B$ is a tautology or 'A tautologically implies B.

Implications :

1. $P \wedge Q \Rightarrow P$.
2. $P \wedge Q \Rightarrow Q$.
3. $P \Rightarrow P \vee Q$
4. $\neg P \Rightarrow P \to Q$
5. $Q \Rightarrow P \to Q$
6. $\neg(P \to Q) \Rightarrow P$.
7. $\neg(P \to Q) \Rightarrow \neg Q$
8. $P \wedge (P \to Q) \Rightarrow Q$
9. $\neg Q \wedge (P \to Q) \Rightarrow \neg P$
10. $\neg P \wedge (P \vee Q) \Rightarrow Q$
11. $(P \to Q) \wedge (Q \to R) \Rightarrow P \to R$
12. $(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \Rightarrow R$.

①

P	Q	$P \wedge Q$	$(P \wedge Q) \to P$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

2.

Implication is transitive:

Proof:

Assume that $A \Rightarrow B$, $B \Rightarrow C$

we have to prove $A \Rightarrow C$

Since $A \Rightarrow B$ and $B \Rightarrow C$

ie, $A \to B$ and $B \to C$ are tautologies

$\therefore (A \to B) \land (B \to C)$ is a tautology.

from formula (11)

$$(P \to Q) \land (Q \to R) \Rightarrow (P \to R)$$

$$(A \to B) \land (A \to C) \Rightarrow A \to C$$

$\therefore A \to C$ is a tautology

$\therefore A \Rightarrow C$

$\therefore$ Implication is transtive

Theorem: 1.2m

If $H_1, H_2 \cdots H_m$ and P imply Q, then $H_1, H_2 \cdots H_m$ imply $P \to Q$.

Proof:

From our assumption we have

$$(H_1 \land H_2 \land \cdots \land H_m \land P) \Rightarrow Q$$

we have to prove

$$H_1 \land H_2 \land \cdots \land H_m \Rightarrow (P \to Q)$$

Since $(H_1 \land H_2 \land \cdots \land H_m \land P) \Rightarrow Q$

$(H_1 \land H_2 \land \cdots \land H_m \land P) \to Q$ is a tautology.

Now using the equivalence relation

$$P_1 \to (P_2 \to P_3) \Leftrightarrow (P \land P_2) \to P_3 \quad [\text{By } P \to (Q \to R) \Leftrightarrow (P \land Q) \to R]$$

$$(P_1 \land P_2) \to P_3 \Leftrightarrow P_1 \to (P_2 \to P_3)$$

we have $(H_1 \land H_2 \land \cdots \land H_m) \to (P \to Q)$ is a tautology.

which implies that

$(H_1 \wedge H_2 \wedge \dots \wedge H_m) \Rightarrow (P \rightarrow Q)$

Hence the proof.

Problems:

Show the following implications

1. $(P \wedge Q) \Rightarrow (P \rightarrow Q)$

P	Q	$P \wedge Q$	$P \rightarrow Q$	$(P \wedge Q) \rightarrow (P \rightarrow Q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

$(P \wedge Q) \rightarrow (P \rightarrow Q)$ is a Tautology

$\therefore (P \wedge Q) \Rightarrow (P \rightarrow Q)$

2. $(P \rightarrow (Q \rightarrow R)) \Rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$

Show the following equivalences.

(a) $(P \rightarrow (Q \rightarrow R)) \Leftrightarrow \neg P \rightarrow (P \rightarrow Q)$

(b) $\neg (P \rightleftarrows Q) \Leftrightarrow ((P \vee Q) \wedge \neg (P \wedge Q))$

Using the statements

R: Mark is rich

H: Mark is happy

Write the following statements in symbolic form

(i) Mark is poor but happy $\neg R \wedge H$

(ii) Mark is rich or unhappy $R \vee \neg H$

(iii) Mark is neither rich nor happy $\neg R \wedge \neg H$.

Construct the truth tables for the following formulas (27)

(a) $\neg(\neg P \vee \neg Q)$ (b) $\neg(\neg P \wedge \neg Q)$ (c) $P \wedge (P \vee Q)$

(d) $P \wedge (P \wedge Q)$ (e) $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

(f) $(P \wedge Q) \vee (\neg P \vee Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$

Give the truth values of P and Q as T and those of R and S as F. Find the truth values of the following

(a) $P \vee (Q \wedge R)$ (b) $(P \wedge (Q \wedge R)) \vee \neg((P \vee Q) \wedge (R \vee S))$

(c) $(\neg(\underline{P \wedge Q}) \vee \neg R) \vee (((\neg P \wedge Q) \vee \neg R) \wedge S)$

P. Q : T
R. S : F.

P	Q	¬Q	R	¬R	S	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg(P \wedge Q) \vee \neg R$
T	T	F	F	T	F	T	F	T

Tautologies ...

which implies that

$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$

$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (P \vee Q) \wedge (\neg P \wedge \neg P) \wedge Q$ [∵ Associative]

$\Leftrightarrow (P \vee Q) \wedge (\neg P \wedge Q)$ [by $P \wedge P \Leftrightarrow P$]

$\Leftrightarrow ((P \vee Q) \wedge \neg P) \wedge Q$ [Associative]

$\Leftrightarrow ((P \wedge \neg P) \vee (Q \wedge \neg P)) \wedge Q$ [Distributive]

$\Leftrightarrow (F \vee (Q \wedge \neg P)) \wedge Q$ [by $P \wedge \neg P \Leftrightarrow F$]

$\Leftrightarrow (Q \wedge \neg P) \wedge Q$ [by $F \vee P \Leftrightarrow P$]

$\Leftrightarrow (\neg P \wedge Q) \wedge Q$

$\Leftrightarrow \neg P \wedge (Q \wedge Q)$ [Associative]

$\Leftrightarrow \neg P \wedge Q$ [Idempotent law]

Normal forms:-

1. Disjunctive normal forms:

5 mark

Defn: Elementary product & Elementary sum:

A product of the variables and their negations in a formula is called an elementary product.

Similarly a sum of the variables and their negations in a formula is called an elementary sum.

Ex:

Let P & Q be any two atomic variables then $P, Q, \neg P \wedge Q, \neg Q \wedge P \wedge \neg P$, $P \wedge \neg P$, $\neg Q \wedge P$, $\neg P \wedge \neg Q$ are some of examples of elementary products.

Let P & Q be any two atomic variables. Then $P, Q, \neg P \vee Q, \neg Q \vee P \vee \neg P$, $P \vee \neg P$, $\neg Q \vee P$, $\neg P \vee \neg Q$ are examples of elementary sums of two variables P & Q.

Factor of elementary sum and Product (29)

Any part of an elementary sum or product which is itself an elementary sum or product is called an original factor of elementary sum or product.

Ex: The factors of $\neg Q \wedge P \wedge \neg Q$ are $\neg Q$, $\neg Q \wedge P$, $P \wedge \neg P$, $\neg P$

1m. "A necessary and sufficient condition for an elementary products to be identically false is that it contains atleast one pair of factor in which one is the negation of other."

1m. • A necessary and sufficient condition for an elementary sum to be identically true is that it contains atleast one pair of factors in which one is the negation of other.

Disjunctive normal forms:-

Defn:

A formula which is equivalent to given formula and which consists of a sum of elementary products is called a disjunctive normal form of the given formula.

DF $(P \wedge \neg P) \vee (P \wedge Q)$

Ex: Obtain Disjunctive normal forms of

(a) $P \wedge (P \to Q)$

Smart (b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

Soln:

since $P \rightleftarrows Q$:

(a) $P \wedge (P \to Q) \Leftrightarrow P \wedge (\neg P \vee Q)$

$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$

(b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

Using $R \rightleftarrows S \Leftrightarrow (R \wedge S) \vee (\neg R \wedge \neg S) \to$ since

$\underset{R}{\neg(P \vee Q)} \rightleftarrows \underset{S}{(P \wedge Q)} \Leftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee (\neg\neg(P \vee Q) \wedge \neg(P \wedge Q))$

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (((P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge \neg Q))$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \wedge \neg P) \vee (Q \wedge \neg P)) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$

which is the required disjunctive normal form.

Tautological ... 30

The disjunctive form of a given formula is not unique i.e. different disjunctive normal forms can be obtained for a given formula, if the distributive laws are applying in different ways.

A given formula is identically false if every elementary product appearing in its normal form is identically false.

Conjunctive normal forms:-

Defn:- A formula which is equivalent to a given formula and which consists a product of elementary sums is called a conjunctive normal form of the given formula.

A given formula is identically true if every elementary sum in its conjunctive normal form is identically true.

this format (V) $\wedge$ (V) $\wedge$ (V)

Ex: Obtain a conjunctive normal form of

(a) $P \wedge (P \to Q)$

(b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

Soln.

(a) $P \wedge (P \to Q) \Leftrightarrow P \wedge (\neg P \vee Q)$

(b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

[using $R \rightleftarrows S \Leftrightarrow (R \to S) \wedge (S \to R)$]

$$\neg(P \vee Q) \rightleftarrows (P \wedge Q) \Leftrightarrow (\neg(P \vee Q) \to (P \wedge Q)) \wedge ((P \wedge Q) \to \neg(P \vee Q))$$

$$\Leftrightarrow (\neg\neg(P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee \neg(P \vee Q))$$

$$\Leftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge ((\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q))$$

$$\Leftrightarrow (((P \vee Q) \vee P) \wedge ((P \vee Q) \vee Q)) \wedge (\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q)$$

$$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\neg P \vee \neg Q \vee \neg Q) \wedge (\neg P \vee \neg Q \vee \neg Q)$$

Ex:

Show that the formula $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a Tautology. (3)

Soln:

First we obtain a conjuntive normal form of the given formula.

$$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) \Leftrightarrow Q \vee ((P \vee \neg P) \wedge \neg Q) \quad \text{distributive}$$
$$\Leftrightarrow (Q \vee (P \vee \neg P)) \wedge (Q \vee \neg Q)$$
$$\Leftrightarrow (Q \vee P \vee \neg P) \wedge (Q \vee \neg Q)$$

Since each of the elementary sums is a tautology, the given formula is a tautology.

SM

Principal Disjunctive Normal Forms: (PDNF) [Sum of minterms]

Let P and Q be two statement variables. Let us construct all possible formulae which consists of conjunctions of P or its negation and conjunctions of Q or its negation none of the formulae should contain both a variable and its negation.

For two variables P and Q there are 2^2 such formulae are given by

1. $P \wedge Q$
2. $P \wedge \neg Q$
3. $\neg P \wedge Q$
4. $\neg P \wedge \neg Q$

These formulas are called minterms or Boolean conjunctions of P and Q.

Each minterm has the truth value T for exactly one combination of the truth values of the variables P & Q

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

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Problem:

Find the principal disjunctive normal form of the following formulas.

$P \to Q$, $P \vee Q$, $\neg(P \wedge Q)$

P	Q	$P \to Q$	$P \vee Q$	$\neg(P \wedge Q)$
T	T	T	T	F
F	T	T	T	T
T	F	F	T	T
F	F	T	F	T

$\therefore (P \to Q) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

$(P \vee Q) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (P \wedge \neg Q)$

$\neg(P \wedge Q) \Leftrightarrow (\neg P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$

The number of entries appearing in the normal form is same as the number of entries with truth value true (T) in the truth table of the given formula. Thus every formula which not a Contradiction has an equivalent PDNF. Such a normal form is unique.

Principal disjunctive normal form for 3 variables: (disjunctive → product of elementary sums)

The minterms for the 3 variables P, Q, R are given by $(P \wedge Q \wedge R)$, $(P \wedge Q \wedge \neg R)$, $(P \wedge \neg Q \wedge R)$, $(\neg P \wedge Q \wedge R)$, $(\neg P \wedge \neg R \wedge R)$, $(P \wedge \neg Q \wedge \neg R)$, $(\neg P \wedge Q \wedge \neg R)$, $(\neg P \wedge \neg Q \wedge \neg R)$.

Here there are 2^3 minterms for 3 variables.
Generally there are 2^n minterms for n variables.

For any formula containing n variables $P_1, P_2 \cdots P_n$ an equivalent disjunctive normal form can be obtained by selecting appropriate minterms out of the set of 2^n possible minterms.

Shri Rajkamal Xerox
Gobi

If a formula is a tautology then all the minterms appear in the principal Disjunctive normal form.

In order to obtain the principal Disjunctive normal form of a given formula without constructing the truth table

* Replace Conditionals & Biconditionals by their equivalent formulae containing only $\vee, \wedge$ & $\neg$.
* The Negations are applied to the variables by using Demorgan's laws followed by the application of distributive laws.
* Any elementary product which is a contradiction is dropped.
* Minterms are obtained in the disjunctions by introducing the missing factors.
* identical minterms appearing in the disjunctions are deleted.

Ex: obtain the PDNF of the following formulae. 8mark

a. $\neg P \vee Q$

b. $(P \wedge Q) \vee (\neg P \vee R) \vee (Q \wedge R)$

Soln:

(a) $(\neg P \vee Q) \Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee ((Q \wedge P) \vee (Q \wedge \neg P))$ → using assoc.

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$

(b) $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$\Leftrightarrow ((P \wedge Q) \wedge (R \vee \neg R)) \vee ((\neg P \wedge R) \wedge (Q \vee \neg Q)) \vee ((Q \wedge R) \wedge (P \vee \neg P))$

$\Leftrightarrow (((P \wedge Q) \wedge R) \vee ((P \wedge Q) \wedge \neg R)) \vee (((\neg P \wedge R) \wedge Q) \vee ((\neg P \wedge R) \wedge \neg Q))$
$\vee (((Q \wedge R) \wedge P) \vee ((Q \wedge R) \wedge \neg P))$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R)$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$

Ex. Show that the following are equivalent formulas.

a. $P \vee (P \wedge Q) \Leftrightarrow P$

b. $P \vee (\neg P \wedge Q) \Leftrightarrow P \vee Q$.

Soln

(a) $P \vee (P \wedge Q) \Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (P \wedge Q)$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (P \wedge Q)$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)$

$\Leftrightarrow P \wedge (Q \vee \neg Q)$

$\Leftrightarrow P$

(b) $P \vee (\neg P \wedge Q) \Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (\neg P \wedge Q)$

$\Leftrightarrow ((P \wedge Q) \vee (P \wedge \neg Q)) \vee (\neg P \wedge Q)$ — ①

$(P \vee Q) \Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$ — ②

from ① & ②.

$P \vee (\neg P \wedge Q) \Leftrightarrow (P \vee Q)$.

Ex: Obtain the principal disjunctive normal form of $P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$.

Soln:

Using $P \rightarrow Q \Leftrightarrow \neg P \vee Q$ and Demorgan's law, we obtain

$P \rightarrow Q \Leftrightarrow \neg P \vee Q$

$P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg(Q \vee \neg P))$

$\Leftrightarrow \neg P \vee ((\neg P \vee Q) \wedge \neg\neg(Q \wedge P))$

$\Leftrightarrow \neg P \vee ((\neg P \vee Q) \wedge (Q \wedge P))$

$\Leftrightarrow \neg P \vee ((\neg P \wedge (Q \wedge P)) \vee (Q \wedge (Q \wedge P)))$

$\Leftrightarrow \neg P \vee (((\underline{\neg P \wedge P}) \wedge Q) \vee ((Q \wedge Q) \wedge P))$

$\Leftrightarrow \neg P \vee (F \wedge Q) \vee (Q \wedge P)$

$\Leftrightarrow \neg P \vee F \vee (Q \wedge P)$

$\Leftrightarrow \neg P \vee (Q \wedge P)$

$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge P)$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P)$

sum of elementary product

Principal Conjuntive Normal form:

For a given number of variables, the maxterms consists of disjunctions in which each variable or its negation but not both appears only once. Thus the maxterms are the duals of minterms.

For the two variables P & Q the max terms are $P \vee Q$, $\neg P \vee Q$, $\neg P \vee \neg Q$, $P \vee \neg Q$.

Defn. (PCNF) **Principal Conjunctive Normal form**

For a given formula an equivalent formula consist of conjunctions of the maxterms only is known as its principal conjunctive normal form.

If the principal disjunctive (conjunctive) normal form of a given formula A containing n variables is known, then the principal disjunctive (conjunctive) normal form of negation of A will consists of the disjuction (conjunction) of the remaining minterms (maxterms) which donot appear in the principal Disjunctive (Conjunctive) Normal form of A.

From the relation $A \Leftrightarrow \neg\neg A$ we obtain the principal Disjunctive (conjunctive) Normal form of A by repeated applications of Demorgan's laws to principal disjunctive (Conjunctive) Normal of $\neg A$.

Ex:1 Obtain the Principal conjunctive Normal form of the formula S is given by $(\neg P \to R) \wedge (Q \rightleftarrows P)$. Also find the Principal disjunctive Normal form of S. (36)

$P \to Q \Rightarrow \neg P \vee Q$

$Q \to P \Rightarrow \neg Q \vee P$

Soln:

$$(\neg P \to R) \wedge (Q \rightleftarrows P)$$

$$\Leftrightarrow (\neg\neg P \vee R) \wedge ((Q \to P) \wedge (P \to Q))$$

$$\Leftrightarrow (P \vee R) \wedge ((\neg Q \vee P) \wedge (\neg P \vee Q))$$

$$\Leftrightarrow ((P \vee R) \vee (Q \wedge \neg Q)) \wedge ((\neg Q \vee P) \vee (R \wedge \neg R)) \wedge ((\neg P \vee Q) \vee (R \wedge \neg R))$$

$$\Leftrightarrow (((P \vee R) \vee Q) \wedge ((P \vee R) \vee \neg Q)) \wedge (((\neg Q \vee P) \vee R) \wedge ((\neg Q \vee P) \vee \neg R)) \wedge (((\neg P \vee Q) \vee R) \wedge ((\neg P \vee Q) \vee \neg R))$$

$$\Leftrightarrow ((P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$

$$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$

which is Principal Conjunctive normal form of S.

To obtain the Disjunctive Normal form of $\neg S$ we have to write the conjunctions of the remaining maxterms.

The principal conjunctive Normal form of $\neg S$ is given by

$$\neg S \Leftrightarrow (P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$$

$$\neg\neg S \Leftrightarrow \neg((P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R))$$

$$\Leftrightarrow \neg(P \vee Q \vee \neg R) \vee \neg(\neg P \vee \neg Q \vee R) \vee \neg(\neg P \vee \neg Q \vee \neg R)$$

$$\Leftrightarrow (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$$

which is the Principal Disjunctive Normal form of S.

Ex:2 Obtain the Principal disjunctive and conjunctive Normal form of $(\neg P \vee \neg Q) \to (P \rightleftarrows \neg Q)$.

Soln:

$$(\neg P \vee \neg Q) \to (P \rightleftarrows \neg Q)$$

$$\Leftrightarrow \neg(\neg P \vee \neg Q) \vee (P \rightleftarrows \neg Q)$$

$$\Leftrightarrow \neg(\neg P \vee \neg Q) \vee ((P \wedge \neg Q) \vee (\neg P \wedge Q))$$

$\Leftrightarrow \neg\neg(P\wedge Q) \vee ((P\wedge \neg Q) \vee (\neg P \wedge Q))$

$\Leftrightarrow (P\wedge Q) \vee (P\wedge \neg Q) \vee (\neg P\wedge Q)$ which is the PDNF of S

The Principal Disjunctive Normal form of $\neg S$ is given by

$\neg S \Leftrightarrow (\neg P \wedge \neg Q)$

$\neg\neg S \Leftrightarrow \neg\neg(P \vee Q)$

$\Leftrightarrow (P\vee Q)$

which is the principal conjunctive Normal Form of S.

1). Find the principal disjunctive and conjunctive Normal form of the following formula.

$(Q\to P)\wedge(\neg P\wedge Q)$

Soln:

$(Q\to P)\wedge(\neg P\wedge Q) \Leftrightarrow (\neg Q \vee P)\wedge(\neg P\wedge Q)$

$\Leftrightarrow (\neg Q\vee P)\wedge(\neg P \vee \underline{(Q\wedge\neg Q)})\wedge(Q\vee\underline{(P\wedge\neg P)})$

$\Leftrightarrow (\neg Q\vee P)\wedge((\neg P\vee Q)\wedge(\neg P\vee\neg Q))\wedge((Q\vee P)\wedge(Q\vee\neg P))$

$\Leftrightarrow (P\vee\neg Q)\wedge(\neg P\vee Q)\wedge(\neg P\vee\neg Q)\wedge(P\vee Q)\wedge(\neg P\vee Q)$

$\Leftrightarrow (P\vee\neg Q)\wedge(\neg P\vee Q)\wedge(\neg P\vee\neg Q)\wedge(P\vee Q)$

which is the <u>Conjunctive</u> Normal form.

There is no principal disjunctive Normal form.

2). Find the product of Sums of the Canonical Forms of the following formula

$(P\wedge Q\wedge R)\vee(\neg P\wedge R\wedge Q)\vee(\neg P\wedge\neg Q\wedge\neg R)$

Soln:

$S \Leftrightarrow (P\wedge Q\wedge R)\vee(\neg P\wedge Q\wedge R)\vee(\neg P\wedge\neg Q\wedge\neg R)$

$\neg S \Leftrightarrow (P\wedge\neg Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(P\wedge\neg Q\wedge\neg R)\vee(\neg P\wedge\neg Q\wedge R)$
$\vee(\neg P\wedge Q\wedge\neg R)$

$\neg\neg S \Leftrightarrow \neg((P\wedge\neg Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(P\wedge\neg Q\wedge\neg R)\vee(\neg P\wedge\neg Q\wedge R)$
$\vee(\neg P\wedge Q\wedge\neg R))$

$\Leftrightarrow \neg(P\wedge\neg Q\wedge R)\wedge\neg(P\wedge Q\wedge\neg R)\wedge\neg(P\wedge\neg Q\wedge\neg R)\wedge\neg(\neg P\wedge\neg Q\wedge R)$
$\wedge\neg(\neg P\wedge Q\wedge\neg R)$

$\Leftrightarrow (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee Q \vee \neg R)$
$\wedge (P \vee \neg Q \vee R)$ $\wedge$ (38)

which is principal conjunctive normal form.

Find the product & sums canonical forms
$(P \wedge Q) \vee (\neg P \wedge Q) \vee (P \wedge \neg Q)$

Soln:

$S \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (P \wedge \neg Q)$

$\neg S \Leftrightarrow (\neg P \wedge \neg Q)$

$\neg\neg S \Leftrightarrow \neg(\neg P \wedge \neg Q)$

$\Leftrightarrow \neg\neg(P \vee Q)$

$\Leftrightarrow (P \vee Q)$.

3. Find PDNF & PCNF & $Q \wedge (P \vee \neg Q)$

Soln

$Q \wedge (P \vee \neg Q) \Leftrightarrow (Q \vee (P \wedge \neg P)) \wedge (P \vee \neg Q)$

$\Leftrightarrow (Q \vee P) \wedge (Q \vee \neg P) \wedge (P \vee \neg Q)$

$S \Leftrightarrow (P \vee Q) \wedge (\neg P \vee Q) \wedge (P \vee \neg Q)$

which is PCNF

$\neg S \Leftrightarrow (\neg P \vee \neg Q)$

$\neg\neg S \Leftrightarrow \neg\neg(P \wedge Q)$

$\Leftrightarrow (P \wedge Q)$

which is PDNF

4. Write the equivalent forms for the following formula in which the negations are applied to the variable only and also obtain the PCNF &

(i) $\neg(P \vee Q)$ (ii) $\neg(P \to Q)$.

Soln:

(i) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

$\Leftrightarrow (\neg P \vee (Q \wedge \neg Q)) \wedge (\neg Q \vee (P \wedge \neg P))$

$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (\neg Q \vee P) \wedge (\neg Q \vee \neg P)$

$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee \neg Q)$

$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (P \vee \neg Q)$.

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(ii) $\neg(P \to Q) \Leftrightarrow \neg(\neg P \vee R)$

$\Leftrightarrow P \wedge \neg Q$

$\Leftrightarrow (P \vee (Q \wedge \neg Q)) \wedge (\neg Q \vee (P \wedge \neg P))$

$\Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg Q \vee P) \wedge (\neg Q \vee \neg P)$

$\Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee \neg Q)$

1. Find the product & sums of canonical forms of the following formulae

(i) $(P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R)$

(ii) $(\neg S \wedge \neg P \wedge R \wedge Q) \vee (S \wedge P \wedge \neg Q \wedge \neg R) \vee (\neg S \wedge P \wedge R \wedge \neg Q) \vee$
$(Q \wedge \neg P \wedge \neg R \wedge S) \vee (P \wedge \neg S \wedge \neg R \wedge Q)$

2. Find the PDNF of $P \to (P \wedge (Q \to P))$

1. (i) $S \Leftrightarrow (P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R)$

$\neg S \Leftrightarrow (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R)$
$\vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

$\neg\neg S \Leftrightarrow \neg((P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R)$
$\vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

By demorgan's law

$S \Leftrightarrow \neg(P \wedge \neg Q \wedge R) \wedge \neg(P \wedge Q \wedge \neg R) \wedge \neg(\neg P \wedge \neg Q \wedge R) \wedge \neg(P \wedge \neg Q \wedge \neg R)$
$\wedge \neg(\neg P \wedge Q \wedge \neg R) \wedge \neg(\neg P \wedge \neg Q \wedge \neg R)$

$S \Leftrightarrow (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R)$
$\wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \vee R)$

which is the required PCNF.

(ii) $A \Leftrightarrow (\neg P \wedge Q \wedge R \wedge \neg S) \vee (P \wedge \neg Q \wedge \neg R \wedge S) \vee (P \wedge \neg Q \wedge R \wedge \neg S)$
$\vee (\neg P \wedge Q \wedge \neg R \wedge S) \vee (P \wedge Q \wedge \neg R \wedge \neg S)$

$\neg A \Leftrightarrow (\neg P \wedge \neg Q \wedge \neg R \wedge \neg S) \vee (\neg P \wedge \neg Q \wedge \neg R \wedge S) \vee (\neg P \wedge Q \wedge \neg R \wedge \neg S)$
$\vee (\neg P \wedge \neg Q \wedge R \wedge \neg S) \vee (P \wedge \neg Q \wedge \neg R \wedge \neg S) \vee (\neg P \wedge \neg Q \wedge R \wedge S)$
$\vee (\neg P \wedge Q \wedge R \wedge S) \vee (P \wedge \neg Q \wedge R \wedge S) \vee (P \wedge Q \wedge \neg R \wedge S) \vee (P \wedge Q \wedge R \wedge \neg S)$
$\vee (P \wedge Q \wedge R \wedge S)$

$\neg\neg A \Leftrightarrow \neg(\neg P \wedge \neg Q \wedge \neg R \wedge \neg S) \wedge \neg(\neg P \wedge \neg Q \wedge \neg R \wedge S) \wedge \neg(\neg P \wedge Q \wedge \neg R \wedge \neg S)$
$\wedge \neg(\neg P \wedge \neg Q \wedge R \wedge \neg S) \wedge \neg(P \wedge \neg Q \wedge \neg R \wedge \neg S) \wedge \neg(\neg P \wedge \neg Q \wedge R \wedge S)$
$\wedge \neg(\neg P \wedge Q \wedge R \wedge S) \wedge \neg(P \wedge \neg Q \wedge R \wedge S) \wedge \neg(P \wedge Q \wedge \neg R \wedge S) \wedge$
$\neg(P \wedge Q \wedge R \wedge \neg S) \wedge \neg(P \wedge Q \wedge R \wedge S)$

$A \Leftrightarrow (P \vee Q \vee R \vee S) \wedge (P \vee Q \vee R \vee \neg S) \wedge (P \vee \neg Q \vee R \vee S) \wedge (P \vee Q \vee \neg R \vee S)$
$\wedge (\neg P \vee Q \vee R \vee S) \wedge (P \vee Q \vee \neg R \vee \neg S) \wedge (P \vee \neg Q \vee \neg R \vee \neg S)$
$\wedge (\neg P \vee Q \vee \neg R \vee \neg S) \wedge (\neg P \vee \neg Q \vee R \vee S) \wedge (\neg P \vee Q \vee R \vee S)$
$\wedge (\neg P \vee \neg Q \vee \neg R \vee \neg S)$

Which is the required PCNF

let $A \Leftrightarrow P \to (P \wedge (Q \to P))$
$\Leftrightarrow \neg P \vee (P \wedge (Q \to P))$
$\Leftrightarrow \neg P \vee (P \wedge (\neg Q \vee P))$
$\Leftrightarrow \neg P \vee ((P \wedge \neg Q) \vee (P \vee P))$
$\Leftrightarrow \neg P \vee ((P \vee P) \vee (P \wedge \neg Q))$ ✓
$\Leftrightarrow (\neg P \vee P) \vee (P \wedge \neg Q)$
$\Leftrightarrow (P \wedge \neg Q)$

$\neg A \Leftrightarrow (\neg P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (P \wedge Q)$

$\neg\neg A \Leftrightarrow \neg(\neg P \wedge \neg Q) \wedge \neg(\neg P \wedge Q) \wedge \neg(P \wedge Q)$

$A \Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee \neg Q)$

Which is required PCNF.

Ex:

The truth table for a formula A is given by following table. Find its disjunctive & Conjunctive normal forms.

P	Q	R	A
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	T

Soln.

$A \Leftrightarrow (P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$ (41).

which is a disjunctive Normal Form.

$A \Leftrightarrow (\neg P \vee \neg Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee Q \vee \neg R)$

Which is Conjunctive normal Form.

Ordering and Uniqueness of Normal Forms:

Any n statement variables be given, first let us arrange them in some fixed orders.

If capital letters are used to denote the variables they may be arranged in alphabetical order.

If subscripted letters are also used then we follow the following illustration.

$A, B, \ldots Z, A_1, B_1 \ldots Z_1, A_2, B_2 \ldots Z_2 \ldots$

If the variables are

$P_1, Q, R_3, S_1, T_2, Q_3$ which can be written as

$Q, P_1, S_1, T_2, Q_3, R_3$.

Once the variables can be arranged in a particular order, it is possible to designate them as the first variable, second variable and so on.

Uniqueness:-

Let us assume that n-variables are given and are arranged in a particular order. The 2^n minterms corresponding to the n-variables can be designated by

$m_0, m_1, \ldots m_{2^n-1}$

If we write the subscript of any particular minterm in binary and add an appropriate number of zeros on the left (if necessary) so that the number of digits in the subscript is exactly n, then we can obtain the corresponding minterms in the following manner

Thus each of

$m_0, m_1, \ldots m_{2^n-1}$ corresponds to a unique minterm which can be determined from the binary representation of the subscript.

Conversely, Given any minterm we can find which of $m_0, m_1, \ldots m_{2^n-1}$ designates it.

Ex:1 let P, Q, R be the three variables arranged in that order.

The corresponding 8 minterms are denoted by $m_0, m_1, \ldots m_7$.

we can write the subscript 5 in binary as 101

The minterm m_0 is $\neg P \wedge \neg Q \wedge \neg R$

The minterm m_3 is $\neg P \wedge Q \wedge R$

The minterm m_5 is $P \wedge \neg Q \wedge R$

2. Consider there are 6 variables.

$P_1, P_2 \ldots P_6$

There are 2^6 possible minterm $= 64$.

$m_0, m_1, \ldots m_{63}$

The minterm corresponding to m_{38} is

100110

$P_1 \wedge \neg P_2 \wedge \neg P_3 \wedge P_4 \wedge P_5 \wedge \neg P_6$

Having developed a notation for the representation of notations, which can be further simplified, by writing only subscripts of $m_0, m_1, \ldots m_{2^n-1}$

we designate the disjunction (sum) of minterms by the compact notation Σ.

Using such a notation the sum of product canonical forms representing the disjunction of m_i, m_j & m_k can be written as $\sum_{i,j,k}$.

Ex: Consider the formula $(P\wedge Q)\vee(\neg P\wedge R)\vee(Q\wedge R)$ find PDNF

$(P\wedge Q)\vee(\neg P\wedge R)\vee(Q\wedge R)$ conj. Thm: 0

$(\neg P\wedge R)\wedge(Q\vee\neg Q)\vee(Q\wedge R)\wedge P\vee\neg P$ 1

(43)

$\Leftrightarrow ((P\wedge Q)\wedge(R\vee\neg R))\vee(\cancel{\neg P\wedge(Q\vee\neg Q)}\wedge R)\vee((P\vee\neg P)\wedge Q\wedge R)$

$\Leftrightarrow (P\wedge Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(\neg P\wedge Q\wedge R)\vee(\neg P\wedge\neg Q\wedge R)\vee(P\wedge Q\wedge R)$

$\Leftrightarrow (P\wedge Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(\neg P\wedge Q\wedge R)\vee(\neg P\wedge\neg Q\wedge R)$

7 6 3 1

The principal disjunctive Normal form is

$\sum_{1,3,6,7}$

which is the Sum of Products Canonical form of given formula.

Ex: $(P\wedge Q)\vee(\neg P\wedge Q\wedge R)$

Soln:

$(P\wedge Q)\vee(\neg P\wedge Q\wedge R)$

$\Leftrightarrow ((P\wedge Q)\wedge(R\vee\neg R))\vee(\neg P\wedge Q\wedge R)$

$\Leftrightarrow (((P\wedge Q)\wedge R)\vee((P\wedge Q)\wedge\neg R))\vee(\neg P\wedge Q\wedge R)$

$\Leftrightarrow (P\wedge Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(\neg P\wedge Q\wedge R)$

$2^2\ 2^1\ 2^0 \quad 2^2\ 2^1\ 2^0 \quad 2^2\ 2^1\ 2^0$

7 6 3

The PDNF is $\sum_{3,6,7}$

Which is the Sum of Products Canonical Form

Product of Sums (PCNF)

We denote the maxterms of n-variables by

$M_0, M_1, \ldots M_{2^n-1}$.

Again the maxterm corresponding to M_j (say), is obtained by writing j in binary appending (joining) the required number of zeros to the left in order to get n-digits of zero appears in the i^{th} location from the left of

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Thus and n

this binary number, then the i^{th} variable appears in the disjunction ($\vee$), when '1' appears in the i^{th} location then the negation of i^{th} variable appears. Thus the binary representation of subscripts uniquely determines the maxterms

Conversely, every binary representation of numbers between 0 and 2^{n-1} determines a maxterm. Then Convection regarding 1 & 0 here is the opposite of what was used for minterms.

Ex: The maxterms corresponding to 3 variables P, Q, R are $M_0, M_1, M_3, M_4 \dots M_7$

$(P \vee Q \vee R)$, $(\neg P \vee Q \vee R)$, $(P \vee \neg Q \vee R)$, $(P \vee Q \vee \neg R)$, $(\neg P \vee \neg Q \vee R)$, $(\neg P \vee Q \vee \neg R)$, $(P \vee \neg Q \vee \neg R)$, $(\neg P \vee \neg Q \vee \neg R)$.

Now we use product (π) to denote the conjunction (product) of maxterms.

The conjunction of maxterms M_i, M_j, M_k can be written as

$$\pi_{i,j,k}.$$

Ex: obtain the PCNF of $(P \wedge Q) \vee (\neg P \wedge R)$

Sol: $(P \wedge Q) \vee (\neg P \wedge R)$

$\Leftrightarrow ((P \wedge Q) \vee \neg P) \wedge ((P \wedge Q) \vee R)$

$\Leftrightarrow ((P \vee \neg P) \wedge (Q \vee \neg P)) \wedge ((P \vee R) \wedge (Q \vee R))$

$\Leftrightarrow (T \wedge (Q \vee \neg P)) \wedge ((P \vee R) \wedge (Q \vee R))$

$\Leftrightarrow (Q \vee \neg P) \wedge (P \vee R) \wedge (Q \vee R)$

$\Leftrightarrow (\neg P \vee Q \vee (R \wedge \neg R)) \wedge (P \vee (Q \wedge \neg Q) \vee R) \wedge ((P \vee \neg P) \vee Q \vee R)$

$\Leftrightarrow (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee Q \vee R) \wedge (P \vee Q \vee R) \wedge (P \vee Q \vee R)$
$\wedge (\neg P \vee Q \vee R)$

$S \Leftrightarrow (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R)$

$$\begin{array}{cccc} 1\ 0\ 0 & 1\ 0\ 1 & 0\ 0\ 0 & 0\ 1\ 0 \\ 2^2\ 2^1\ 2^0 & 2^2\ 2^1\ 2^0 & 2^2\ 2^1\ 2^0 & 2^2\ 2^1\ 2^0 \\ 4 & 5 & 0 & 2 \end{array}$$

product of sum of canonical form is

$\Pi_{0,2,4,5}$

$\neg S \Leftrightarrow \cancel{(P \vee \neg Q \vee R)} \wedge (P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$

$\neg\neg S \Leftrightarrow \cancel{\neg(P \vee \neg Q \vee R)} \vee \neg(P \vee Q \vee \neg R) \vee \neg(P \vee \neg Q \vee \neg R) \vee \neg(\neg P \vee \neg Q \vee R) \vee \neg(\neg P \vee \neg Q \vee \neg R)$

$\Leftrightarrow \cancel{(\neg P \wedge Q \wedge \neg R)} \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$

$$\begin{array}{cccc} 0\ 0\ 1 & 0\ 1\ 1 & 1\ 1\ 0 & 1\ 1\ 1 \end{array}$$

Sum of product of canonical Form

$\Sigma_{1,3,6,7}$.

1. Find the Product of Sums Canonical Forms of the following formula.

$(P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$.

<u>Soln</u>:

let $S \Leftrightarrow (P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

$$\begin{array}{ccc} 1\ 1\ 1 & 0\ 1\ 1 & 0\ 0\ 0 \end{array}$$

The Sum of product is

$\Sigma_{0,3,7}$

$\neg S \Leftrightarrow (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R)$

$\neg\neg S \Leftrightarrow (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R)$

$$\begin{array}{ccccc} 1\ 0\ 1 & 1\ 1\ 0 & 0\ 0\ 1 & 0\ 1\ 0 & 1\ 0\ 0 \end{array}$$

The product of Sum cannonical form is

$\Pi_{1,2,4,5,6}$

Ex:

$(P\wedge Q)\vee(\neg P\wedge Q)\vee(P\wedge\neg Q)$ 48

let $S \Leftrightarrow (P\wedge Q)\vee(\neg P\wedge Q)\vee(P\wedge\neg Q)$

The Sum & Product Cannonical Form is $\Sigma_{1,2,3}$

$\neg S \Leftrightarrow \neg P\wedge\neg Q$

$\neg\neg S \Leftrightarrow P\vee Q$

The product & Sums cannonical Form is π_0...

1. Find the product & Sums Cannonical Forms & the following formulae

(1) $(P\wedge Q\wedge R)\vee(\neg P\wedge Q\wedge R)$

(2) $(\neg S\wedge\neg P\wedge R\wedge Q)\vee(S\wedge P\wedge\neg R\wedge\neg Q)\vee(\neg S\wedge P\wedge R\wedge Q)\vee$
$(Q\wedge\neg P\wedge\neg R\wedge S)\vee(P\wedge\neg S\wedge\neg R\wedge Q)$

2. Find PDNF & PCNF &

$P\rightarrow(P\wedge(Q\rightarrow P))$

Soln.

1) (i) $S \Leftrightarrow (P\wedge Q\wedge R)\vee(\neg P\wedge Q\wedge R)$

$\neg S \Leftrightarrow (P\wedge\neg Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(\neg P\wedge\neg Q\wedge R)\vee(P\wedge\neg Q\wedge\neg R)$
$\vee(\neg P\wedge Q\wedge\neg R)\vee(\neg P\wedge\neg Q\wedge\neg R)$

$\neg\neg S \Leftrightarrow \neg((P\wedge\neg Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(\neg P\wedge\neg Q\wedge R)\vee(P\wedge\neg Q\wedge\neg R)$
$\vee(\neg P\wedge Q\wedge\neg R)\vee(\neg P\wedge\neg Q\wedge\neg R))$

$\Leftrightarrow \neg(P\wedge\neg Q\wedge R)\wedge\neg(P\wedge Q\wedge\neg Q)\wedge\neg(\neg P\wedge\neg Q\wedge R)\wedge\neg(P\wedge\neg Q\wedge\neg R)$
$\wedge\neg(\neg P\wedge Q\wedge\neg R)\wedge\neg(\neg P\wedge\neg Q\wedge\neg R)$

$\Leftrightarrow (\neg P\vee Q\vee\neg R)\wedge(\neg P\vee\neg Q\vee R)\wedge(P\vee Q\vee\neg R)\wedge(\neg P\vee Q\vee R)$
$\wedge(P\vee\neg Q\vee R)\wedge(P\vee Q\vee R)$.

which is the PCNF.

(2) $A \Leftrightarrow (\neg P\wedge Q\wedge R\wedge\neg S)\vee(P\wedge\neg Q\wedge\neg R\wedge S)\vee(P\wedge\neg Q\wedge R\wedge\neg S)\vee$
$(\neg P\wedge Q\wedge\neg R\wedge S)\vee(P\wedge Q\wedge\neg R\wedge\neg S)$

$\neg A \Leftrightarrow (\neg P\wedge\neg Q\wedge R\wedge S)\vee(\neg P\wedge Q\wedge R\wedge\neg S)\vee(\neg P\wedge\neg Q\wedge R\wedge S)\vee$
$(\neg P\wedge\neg Q\wedge\neg R\wedge S)\vee(\neg P\wedge\neg Q\wedge R\wedge\neg S)\vee(\neg P\wedge Q\wedge\neg R\wedge\neg S)$
$\vee(\neg P\wedge Q\wedge R\wedge S)\vee(P\wedge\neg Q\wedge R\wedge S)\vee(P\wedge Q\wedge\neg R\wedge S)\vee$
$(P\wedge Q\wedge R\wedge\neg S)\vee(\neg P\wedge\neg Q\wedge\neg R\wedge\neg S)$

The predicate Calculus

Predicate logic:

The logic based upon the analysis of predicates in any statement is called predicates logic.

Predicates:

Ex: John is a bachelor.

Smith is a bachelor.

Both are statements about two different individuals who are bachelors. If we introduce some symbol to denote "is a bachelor" and a method to join it with symbols denoting the names of individuals. Then we will have a symbolism to denote statements about any individuals being a bachelor. That part "is a bachelor" is called "predicate".

Note: we shall symbolize a predicates by a capital letter and the names of individuals or objects in general by small letters.

Consider the statements

1. John is a bachelor
2. Smith is a bachelor

Denote the predicate "is a bachelor" symbolically by the predicate letter 'B'. John by j, and simith by s

(Then the statements ① & ② can be written as $B(j)$ and $B(s)$ respectively)

In general any statement of the type "p is Q where Q is predicate and P is the subject can be denoted by $Q(P)$.

Numbering

(48)

M- place Predicate

A Predicate requiring m (m>0) names is called an "m- place predicate"

Eg: Consider the statements involving the names of two objects, such as

1. Jack is taller than Jill
2. Canada is the north of the United States

The predicates "is taller than" & "is the north of" are 2-place predicates because names of two objects are needed to complete a statement involving these predicates.

If the letter G symbolizes "is taller than" j_1 denotes "Jack" and j_2 denotes "Jill" then the statement ① can be translated as $G(J_1, J_2)$

Similarly, if N denotes the predicates "is to the north of" C: Canada and S: United States then ② can be translated as $N(C, S)$

Examples of 3-place predicates and 4-place predicates

① Susan sits between Ralph and Bill

② Green and Hitter played bridge against Johnson and Smith

In general n-place predicate requires n-names of objects to be inserted in fixed positions in order to obtain a statement

The position of these names is important

If S is an n-place predicate letter and $a_1, a_2 \dots a_n$ are the names of objects, then $S(a_1, a_2 \dots a_n)$ is a statement.

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The Statement Function, Variables and Quantifiers:

Let H be the predicate "is a mortal" b the name "Mr. Benson", c "Canada" and s "Athist".

Then H(b), H(c) and H(s) all denote statements, Infact these statements have a common form.

If we write H(x) for "x is mortal" then H(b), H(s), H(c) and others having the same form can be obtained from H(x) by replacing x by an appropriate name.

Note that H(x) is not a statement, but it results in a statement when x is placed by the name of an object

The letter x used here is a placeholder.

Simple Statement Function:

A Simple Statement function of one variables is defined to be an expression consisting of a predicates symbol and an individual variable.

Such a statement function becomes a statement when the variable is replaced by the name of any object.

The statement resulting from a replacement is called a Substitution instance of the statement function and is a formula of statement calculus.

Compound Statement Function:

For Ex:

if we let M(x) be "x is a man" and H(x) be "x is a mortal" then we can form compound statement functions such as $M(x) \wedge H(x)$ $H(x) \rightarrow H(x)$ $\neg H(x)$ $M(x) \vee \neg H(x)$ etc.

The Statement function of two variable

1. $G(x,y)$: x is taller than y.

If both x & y are replaced by the names of objects we get a statement.

If m represents Mr. Miller and f Mr. Fox, then we have.

$G(m,f)$: Mr. Miller is taller than Mr. Fox

$G(f,m)$: Mr. fox is taller than Mr. Miller

It is not possible to form statement functions of two variables by using statement functions of one variable..

For ex:

Given $M(x)$: x is a man

Universal quantifiers:

Let us first consider the following statements each one is a statement about all individuals or objects belonging to a certain set.

1. All men are mortal
2. Every apple is red
3. Any integer is either positive or negative

1a. For all x, if x is a man, then x is a mortal

2a. For all x, if x is an apple, then x is red

3a. For all x, if x is an integer, then x is either positive or negative

we symbolize "for all x" by the symbols "$(\forall x)$" or by "(x)" using these symbols,

$M(x)$: x is a man $H(x)$: x is a mortal

$A(x)$: x is an apple $R(x)$: x is red

$N(x)$: x is an integer $P(x)$: x is either positive or negative

then we can write

$(x)\,(M(x) \to H(x))$

$(x)\,(A(x) \to R(x))$

$(x)\,(N(x) \to P(x))$

Sometimes $(x)(M(x) \to H(x))$ is also written as $(\forall x)(M(x) \to H(x))$

The symbols (x) or $(\forall x)$ are called universal quantifiers

The quantification symbol is "( )" or "($\forall$)" and it contains the variable which to be quantified

Thus $(x)M(x)$ is a statement which can be translated as, For all x, x is a man

For every x, x is a man

Everything is a man.

Note:

The statement $(x)(M(x) \to H(x))$ and $(y)\,(M(y) \to H(y))$ are equivalent

Sometimes it is necessary to use more than one universal quantifier in a statement

Ex:

consider

$G(x,y)$: x is taller than y.

we can state that "For any x and y, if x is taller than y is not taller than x" (or)

For any x and y, if x is taller than y then it is not true that y is taller than x".

This can be symbolized as

$(x)(y)\,(G(x,y)) \to \neg G(y,x)$.

The universal quantifier was used to translate expressions such as "for all", "every", and "for any".

Existential quantifier:-

Consider the statements.

1. There is a man
2. Some men are clever
3. Some real no's are rational

It can be written as,

1a. There exists an x such that x is a man

1b. There is atleast one x such that x is a man

2a. There exist an x, such that x is a man and x is clever.

2b. There exists atleast one x such that x is a man and x is clever

We use the symbol "$(\exists x)$" called the existential quantifier which symbolizes as

- "There is atleast one x such that" (or)
- "there exists an x such that" (or)
- "for some x" writing

$M(x)$: x is a man

$C(x)$: x is clever

$R_1(x)$: x is real number

$R_2(x)$: x is rational

Using the existential quatifiers

① to ③ can be written as,

$(\exists x)\ (M(x))$

$(\exists x)\ (M(x) \wedge C(x))$

$(\exists x)\ (R_1(x) \wedge R_2(x))$

Predicate Formulae

A well formed formula of Predicate is obtained by using the following Rules

1. An atomic formula is a well formed formula
2. If A is well-formed formula then $\rceil A$ is well formed formula
3. If A & B are well-formed formula then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \rightleftarrows B)$ are also well-formed formula
4. If A is well-formed formula and x is any variable then $(x) A$ and $(\exists x) A$ are well-formed formula
5. Only those formula obtained by using rules ① to ⑤ are well-formed formula } ⑦ sm

Free & Bound Variables:-

Bound occurrence:-

Given a formula containing a part of the form $(x) P(x)$ or $(\exists x) P(x)$. Such a part is called an x-bound part of the formula. Any occurrence of x in an x-bound part of a formula is called a bound occurrence of x

Free occurrence:-

which any occurrence of x or of any variable that is not a bound occurrence is called free occurrence

Scope of the quantifier:-

The formula $P(x)$ either in $(x)P(x)$ or in $(\exists x)P(x)$ is described as the scope of the quantifier.

In otherwords, the scope of quantifier is the formula immediately following the quantifier. If the scope is an atomic formula then no paranthese are used to enclose the formula. otherwise paranthesis are needed.

Ex:

Consider the following formulae

1. $(x)\, P(x, y)$
2. $(x)\, (P(x) \to Q(x))$
3. $(x)\, (P(x) \to (\exists y)\, R(x, y))$
4. $(x)\, (P(x) \to R(x)) \lor (x)\, (P(x) \to Q(x))$
5. $(\exists x)\, (P(x) \land Q(x))$

In ①. $P(x, y)$ is the scope of the quantifier. occurence of x are bound occurence

occurence of y are free occurence.

In ②, Scope of the universal quantifier is $P(x) \to Q(x)$

occurence of x are bound occurences.

In ③, Scope of (x) is $P(x) \to (\exists y)\, R(x, y)$.

while the scope of $(\exists y)$ is $R(x, y)$.

All occurences of both x & y are bound occurences

In ④, Scope of 1st quantifier ——

2nd quantifier ——

All occurence of ~~all~~ x are Bound occurences.

In ⑤, the Scope of $(\exists x)$ is $P(x) \land Q(x)$

Ex: Let $P(x)$: x is a Person

$F(x, y)$: x is the father of y.

$M(x, y)$: x is the mother of y

write the predicate "x is the father of the mother of y".

Soln.

Assume that z as the mother of y and x is the father of z & z as the mother of y. If such a person z exists we symbolize the predicate as

$$(\exists z)\,(P(z) \wedge F(x,z) \wedge M(z,y)).$$

Ex:2 Symbolize the expression "All the world loves a lover".

U-Soln: The quotation really means that everybody loves lover.

Now, let $P(x)$: x is a person
$L(y)$: y is a lover
$R(x,y)$: x loves y.

The required expression is

$$(x)\,(P(x) \rightarrow (y)\,(P(y) \wedge L(y) \rightarrow R(x,y))).$$

5m **Rules of Inference:-**

Or 8m The process of derivation by which one demonstrates that a particular formula is a valid formula is a valid consequence of a given set of premises

For that, define two rules of inference which are called rules P and T.

Rule P: A premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is tautologically implied by any one or more of the preceding formulae in the derivation.) 5m or 8m

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Ex: 1

Demonstrate that R is a valid inference from the premises $P \to Q$, $Q \to R$ and P

Soln

{1}	(1)	$P \to Q$	Rule P
{2}	(2)	P	Rule P
{1, 2}	(3)	Q	Rule T, (1), (2), I_{11}
{4}	(4)	$Q \to R$	Rule P
{1, 2, 4}	(5)	R	Rule T (3), (4) & I_{11}

$(P \to Q) \wedge P$
$\to \neg P \vee Q \wedge P \Rightarrow \neg P \wedge P \vee Q$
$\Rightarrow F \vee Q$
$\to Q$

$\to Q \wedge (Q \to R)$
$\Rightarrow Q \wedge \neg Q \vee R$
$\Rightarrow F \vee R$

Ex: 2 S.T $R \vee S$ follow logically from the premises $\Rightarrow R$
$C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$ and $(A \wedge \neg B) \to (R \vee S)$.

Soln :-

{1}	(1)	$((C \vee D) \to \neg H)$	Rule P
{2}	(2)	$\neg H \to (A \wedge \neg B)$	Rule P
{1, 2}	(3)	$(C \vee D) \to (A \wedge \neg B)$	Rule T (1), (2) and I_{13}
{4}	(4)	$(A \wedge \neg B) \to (R \vee S)$	Rule P
{1, 2, 4}	(5)	$(C \vee D) \to (R \vee S)$	Rule T (3), (4) & I_{13}
{6}	(6)	$C \vee D$	Rule P
{1, 2, 4, 6}	(7)	$R \vee S$	Rule T, (5), (6) & I_{11}

Ex: 3 S.T $S \vee R$ is tautologically implied by
$(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$

Soln :

{1}	(1)	$P \vee Q$	Rule P
{1}	(2)	$\neg P \to Q$	Rule T, (1), E_1 & E_{16}
{3}	(3)	$Q \to S$	Rule P
{1, 3}	(4)	$\neg P \to S$	Rule T, (2), (3) & I_{13}

$\{1,3\}$	(5)	$\neg S \to P$	Rule T, (4), E_{18} & E_1
$\{6\}$	(6)	$P \to R$	Rule P
$\{1,3,6\}$	(7)	$\neg S \to R$	Rule T (5)(6) & I_{13}
$\{1,3,6\}$	(8)	$S \vee R$	Rule T, E_{16} & E_1

Ex: 4 S.T $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ and $\neg M$.

Soln:

$\{1\}$	(1)	$P \to M$	Rule P
$\{2\}$	(2)	$\neg M$	Rule P
$\{1,2\}$	(3)	$\neg P$	Rule T, (1)(2) & I_{12}
$\{4\}$	(4)	$P \vee Q$	Rule P
$\{1,2,4\}$	(5)	Q	Rule T, (3), (4) & I_{10}
$\{6\}$	(6)	$Q \to R$	Rule P
$\{1,2,4,6\}$	(7)	R	Rule T, (5), (6), I_{11}
$\{1,2,4,6\}$	(8)	$(R \wedge (P \vee Q))$	Rule T, (4), (7) & I_9

Ex: 5 S.T I_{12} : $\neg Q$, $P \to Q \Rightarrow \neg P$

Soln:

$\{1\}$	(1)	$P \to Q$	Rule P
$\{1\}$	(2)	$\neg Q \to \neg P$	Rule (1) & E_{18}
$\{3\}$	(3)	$\neg Q$	Rule P
$\{1,3\}$	(4)	$\neg P$	Rule T, (2), (3) & I_{11}

Rule CP: If we can drive S from R and a set of premises, then we can drive, $R \to S$ from the set of premises alone

CP $\to$ Conditional proof.

Rule CP is ~~alone~~ also called the deduction theorem and is generally used if the conclusion is of the form $R \to S$.

In such a cases, R is taken as an additional premise and S is derived from the given premises and R.

Ex : 6 S.T $R \to S$ can be derived from the premises $P \to (Q \to S)$, $\neg R \vee P$ and Q.

Soln :

First ~~order~~ include R as an additional premise

{1}	(1)	$\neg R \vee P$	Rule P
{2}	(2)	R	Rule P (assumed premise)
{1, 2}	(3)	P	Rule T (1), (2), & I_{10}
{4}	(4)	$P \to (Q \to S)$	Rule P
{1, 2, 4}	(5)	$Q \to S$	Rule T, (3), (4) & I_{11}
{6}	(6)	Q	Rule P
{1, 2, 4, 6}	(7)	S	T, (5), (6) & I_{11}
{1, 4, 6}	(8)	$R \to S$	CP

Consistancy of Premises and Indirect method of proof:

Consistent: (consistent)

A set of formulae $H_1, H_2 \cdots H_m$ is said to be consistent if their conjunction has the truth value T for some assignment of the truth values to the atomic variables appearing in $H_1, H_2 \cdots H_m$.

Inconsistent:

If for every assignment of the truth value to the atomic variables, atleast one of the formulae $H_1, H_2 \cdots H_m$ is false so that their conjunction is identically false, then the formulae $H_1, H_2 \cdots H_m$ are called inconsistent.

Note:

A set of formulae $H_1, H_2 \cdots H_m$ is inconsistent if their conjunction implies a contradiction that is,

$$H_1 \wedge H_2 \wedge \cdots \wedge H_m \Rightarrow R \wedge \neg R$$

where R is any formula.

Note that $R \wedge \neg R$ is a contradiction, and it is necessary and sufficient for the implication that $H_1 \wedge H_2 \wedge \cdots \wedge H_m$ be a contradiction.

The notation of the inconsistancy is used in the procedure called proof by contradiction (or) reduction and absurdum or indirect-method of proof.

Ex:1 S.T $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$

Soln: we introduce $\neg\neg(P \wedge Q)$ as an additional premise and show that this additional premise leads to a contradiction.

Rule CP: ¬P ....

{1}	(1)	$\neg\neg(P \land Q)$	Rule P assumed
{1}	(2)	$P \land Q$	Rule T, (1) & E,
{1}	(3)	P	Rule T, (2) & I,
{4}	(4)	$\neg P \land \neg Q$	Rule P
{4}	(5)	$\neg P$	Rule T, (4) & I,
{1, 4}	(6)	$P \land \neg P$	Rule T, (3), (5), I_9

Ex: 2 S. T the following premises are inconsistent

1. If Jack misses many classes through illness then he fails high school
2. If Jack fails high school, then he is uneducated
3. If Jack reads a lot of books, then he is not uneducated
4. Jack misses many classes through illness and reads a lot of books

Soln:

E : Jack misses many classes

S : Jack fails high school

A : Jack reads a lot of books

H : Jack is uneducated.

The premises are $E \to S$, $S \to H$, $A \to \neg H$, and $E \land A$

{1}	(1)	$E \to S$	Rule P
{2}	(2)	$S \to H$	Rule P
{1,2}	(3)	$E \to H$	Rule T, (1), (2) & I_{13}
{4}	(4)	$A \to \neg H$	Rule P
{4}	(5)	$H \to \neg A$	Rule T, (4), E_{18}
{1,2,4}	(6)	$E \to \neg A$	Rule T, (3), (5), I_{13}

$\{1,2,4\}$	(7)	$\neg E \vee \neg A$	Rule T, (6), E_{16}
$\{1,2,4\}$	(8)	$\neg(E \wedge A)$	Rule T, (7), E_8
$\{9\}$	(9)	$E \wedge A$	Rule P
$\{1,2,4,9\}$	(10)	$(E \wedge A) \wedge \neg(E \wedge A)$	Rule T, (8), (9), I_9

Theory of Inference for predicate Calculus:-

The method of derivation involving predicate formulae uses the rules of inference given for the statement calculus and also certain additional rules which are required to deal with the formulae involving quantifiers.

The rules P and T, regarding the introduction of a premise at any stage of derivation and the introduction of any formula which follows logically from the formulae already introduced.

If the conclusion is given in the form of a conditional, we shall also use the rule of conditional proof called CP.

In order to use the equivalences and implications we need some rules on how to eliminate quantifiers during the course of derivation. This elimination is done by rules of specification called rules US & ES

once the quantifiers are eliminated, the derivation proceeds as in the case of the statement calculus, and the conclusion is reached. It may happen that the desired conclusion is quantified

In this case, we need rules of generalisation called rules UG and EG.

Rule CP:

Rules US: Universal Specification:

From $(x)\,A(x)$ one can conclude $A(y)$.

Rules ES: Existential Specification:

From $(\exists x)A(x)$ one can conclude $A(y)$ provided that y is not free in any given premise and also not free in any prior step of the derivation. These requirements can easily be met by choosing a new variable each time ES is used.

The conditions of ES are more restrictive than ordinarily required, but they do not affect the possibility of deriving any conclusion.

Rule EG: Existential Generalization:

From $A(x)$ one can conclude $(\exists y)A(y)$

Rule UG: Universal Generalisation:

From $A(x)$ one can conclude $(y)A(y)$ provided that x is not free in any of the given premises and provided that if x is free in a prior step which resulted from use of ES, then no variables introduced by that use of ES appear free in $A(x)$.

Problems:

Ex:1 S.T $(x)(H(x) \to M(x)) \wedge H(s) \Rightarrow M(s)$. Note that this problem is a symbolic translation of a well-known argument known as the "Socrates argument which is given by

All men are mortal

Socrates is a man

Therefore Socrates is a mortal

If we denote $H(x)$: x is a man,
$M(x)$: x is a mortal

and s : Socrates.

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Soln:

{1}	(1)	$(x)(H(x) \to M(x))$	Rule P
{1}	(2)	$H(s) \to M(s)$	US, (1)
{3}	(3)	$H(s)$	Rule P
{1, 3}	(4)	$M(s)$	Rule T, (2), (3), I_{11}

[In step 2, we remove the universal quantifier]

Ex: 2 S.T $(x)(P(x) \to Q(x)) \wedge (x)(Q(x) \to R(x)) \Rightarrow (x)(P(x) \to R(x))$

Soln:

{1}	(1)	$(x)(P(x) \to Q(x))$	Rule P
{1}	(2)	$P(y) \to Q(y)$	US, (1)
{3}	(3)	$x(Q(x) \to R(x))$	Rule P
{3}	(4)	$Q(y) \to R(y)$	US, (3)
{1, 3}	(5)	$P(y) \to R(y)$	Rule T, (2), (4), I_{13}.
{1, 3}	(6)	$(x)(P(x) \to R(x))$	UG, (5).

Ex: 3 S.T $(\exists x) M(x)$ follows logically from the premise $(x)(H(x) \to M(x))$ & $(\exists x) H(x)$

Soln

{1}	(1)	$(\exists x) H(x)$	Rule P
{1}	(2)	$H(y)$	ES, (1)
{3}	(3)	$(x)(H(x) \to M(x))$	Rule P
{3}	(4)	$H(y) \to M(y)$	US, (3)
{1, 3}	(5)	$M(y)$	Rule T, (2), (4), I_{11}
{1, 3}	(6)	$(\exists x) M(x)$	EG, (5).

Note that, in step 2, the variable y is introduced by ES.

Rule ...

$\therefore$ A conclusion such as $(x)M(x)$ could not follow from step 5 because it would violate the rules given for M_1.

Ex:4 P.T $(\exists x)(P(x) \land Q(x)) \Rightarrow (\exists x)P(x) \land (\exists x)Q(x)$

Soln:

V.Q

$\{1\}$	(1)	$(\exists x)(P(x) \land Q(x))$	Rule P
$\{1\}$	(2)	$P(y) \land Q(y)$	ES, (1), y fixed
$\{1\}$	(3)	$P(y)$	Rule T, (2), I_1
$\{1\}$	(4)	$Q(y)$	Rule T, (2), I_2
$\{1\}$	(5)	$(\exists x)P(x)$	EG, (3)
$\{1\}$	(6)	$(\exists x)Q(x)$	EG, (4)
$\{1\}$	(7)	$(\exists x)P(x) \land (\exists x)Q(x)$	Rule T, (4), (5), I_9

It is instructive to try to prove the converse which does not hold the derivation is

(1)	$(\exists x)P(x) \land (\exists x)Q(x)$	Rule P
(2)	$(\exists x)P(x)$	Rule T, (1), I_1
(3)	$(\exists x)Q(x)$	Rule T, (1), I_2
(4)	$P(y)$	ES, (2)
(5)	$Q(z)$	ES, (3)

(4) y is fixed and it is no longer possible to use that variable again in step 5.

Ex:5 S.T from

(a) $(\exists x)\ (F(x) \wedge S(x)) \rightarrow (y)\ (M(y) \rightarrow W(y))$

(b) $(\exists y)\ (M(y) \wedge \neg W(y))$

the Conclusion $(x)\ (F(x) \rightarrow \neg S(x))$ follows

Soln.

$\{1\}$	(1)	$(\exists y)\ (M(y) \wedge \neg W(y))$	Rule P
$\{1\}$	(2)	$M(z) \wedge \neg W(z)$	ES, (1)
$\{1\}$	(3)	$\neg (M(z) \rightarrow W(z))$	Rule T, (2), E_{17}
$\{1\}$	(4)	$(\exists y)\ \neg(M(y) \rightarrow W(y))$	EG, (3)
$\{1\}$	(5)	$\neg (y)\ (M(y) \rightarrow W(y))$	E_{26}, (4)
$\{6\}$	(6)	$(\exists x)(F(x) \wedge S(x)) \rightarrow (y)\ (M(y) \rightarrow W(y))$	Rule P
$\{1,6\}$	(7)	$\neg (\exists x)\ (F(x) \wedge S(x))$	Rule T, (5), (6), I_{12}
$\{1,6\}$	(8)	$(x)\ \neg (F(x) \wedge S(x))$	Rule T, (7), E_{25}
$\{1,6\}$	(9)	$\neg (F(x) \wedge S(x))$	US, (8)
$\{1,6\}$	(10)	$F(x) \rightarrow \neg S(x)$	Rule T, (9), E_9, E_{16}, E_{17}
$\{1,6\}$	(11)	$(x)\ (F(x) \rightarrow \neg S(x))$	UG, (10)

Ex: 7

"If there was a ball game, then travelling was difficult If they arrived on time, the travelling was not difficult They arrived on Time Therefore there was no ball game".

Show that statements constitutes a valid argument

Soln -

let

P: there was a ball game

Q: Travelling was difficult

R. They arrived on time

we are required to show that from the premises $P \to Q$, $R \to \neg Q$ and R the Conclusion $\neg P$ follows.

Ex: 8

If A works hard, then either B or C will enjoy themselves. If B enjoy himself, then A will not work hard. If D enjoy himself, then C will not.

$\therefore$ If A works hard, D will not enjoy himself.

Soln:

let A : A works hard

B : B will enjoy himself

C : C will enjoy himself

D : D will enjoy himself.

S.T $A \to \neg D$ follows from $A \to B \vee C$, $B \to \neg A$, and $D \to \neg C$. (Hint A as an additional premise)

Consistency of Premises and Indirect method of Proof:-

Consistent :-

A set of formulae $H_1, H_2 \ldots H_m$ is said to be consistent if their conjunction has the truth value T for some assignment of the truth values to the atomic variables appearing in $H_1, H_2 \ldots H_m$.

Inconsistent :